The simplex algorithm (G. Dantzig, 1947)

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Introduction

The algorithm is structured in two phases

Phase 1. Find a BFS or conclude the LP problem is infeasible

Phase 2. Input: a BFS
Output: an optimal BFS or conclude the problem is unbounded

↳ Phase 2 is used as a subroutine in phase 1. We describe phase 2 first

Simplex algorithm - Phase 2

Input: a BFS

a) Optimality test

b) Compute a new BFS that improves the cost and test of unboundedness

Optimality test

Reference LP

$$\max_{x \in X} c^T x \qquad (LP\text{-}S)$$

$$X = \{x \in \mathbb{R}^n : Ax = b, \ x \geq 0\}$$

Pbl. Let B be a feasible basis and consider the vertex $\bar{x}$ given by $\bar{x}_B = B^{-1}b$ and $\bar{x}_F = 0$. Does $\bar{x}$ verify $c^T\bar{x} \geq c^T x$, $\forall x \in X$?

Parametrization of $x \in X$ using B

$$Ax = b \Leftrightarrow B x_B + F x_F = b \longleftrightarrow x_B = B^{-1} b - B^{-1} F x_F$$

↳ dependent variables

Then

$$x \in X \Leftrightarrow \begin{cases} Ax = b \\ x \geq 0 \end{cases} \Longleftrightarrow \begin{cases} B^{-1} b - B^{-1} F x_F \geq 0 \\ x_F \geq 0 \end{cases} \qquad (F)$$

Rmk. x_F are parameters describing all points in X. If $x_F = 0$ one gets the BFS, i.e. the vertex associated to B

Cost of a point $x \in X$

components of the cost associated to B and F

$$c^T x = c_B^T x_B + c_F^T x_F = c_B^T B^{-1} b - c_B^T B^{-1} F x_F + c_F^T x_F =$$

$$= \underbrace{c_B^T B^{-1} b}_{z} + \underbrace{\left(c_F^T - c_B^T B^{-1} F \right)}_{r_F^T} x_F$$

z: cost of the BFS (where $x_F = 0$)

r_F: vector of reduced costs associated to B

Lemma (optimality test). Let B be a feasible basis. If

$$\left.\begin{array}{ll} r_F \le 0 & \text{for a "max" problem} \\ r_F \ge 0 & \text{for a "min" problem} \end{array}\right\} \quad (T)$$

then, the BFS $\bar{x}_F = 0$, $\bar{x}_B = B^{-1} b \ge 0$ is optimal

Rmk. (T) are just sufficient conditions

Corollary. If B is a feasible nondegenerate basis, (T) are necessary and sufficient for optimality

Simplex tableau

We associate to an LP the following equations in the variables $x \in \mathbb{R}^n$ and $z \in \mathbb{R}$

$$\begin{aligned} z + c^T x &= 0 \\ A x &= b \end{aligned} \longrightarrow \begin{bmatrix} 1 & c^T \\ 0 & A \end{bmatrix} \begin{bmatrix} z \\ x \end{bmatrix} = \begin{bmatrix} 0 \\ b \end{bmatrix} \qquad (L)$$

Rmk. $-z$ stores the cost evaluated at x

Tableau associated to (L)

z is already a dependent variables

		z	$x_1 \dots x_n$
z	0	1	c^T
	b	0	A

This column will be not affected by pivoting on elements of A and it can be omitted

Simplex tableau

$$\begin{array}{c|ccc} & x_1 & \cdots & x_n \\ \hline 0 & & c^T & \\ \hline b & & A & \end{array}$$

← row 0

After putting the tableau in the canonical form w.r.t. B, one obtains (up to a permutation of row and columns labeled with variables "x_i")

$$\begin{array}{c|c|ccc|ccc} & & x_{B,1} & \cdots & x_{B,m} & x_{F,1} & \cdots & x_{F,n-m} \\ \hline & z & & 0 & & & r_F^T & \\ \hline x_{B,1} & & & & & & & \\ \vdots & B^{-1}b & & I & & & B^{-1}F & \\ x_{B,m} & & & & & & & \end{array}$$

Opposite of the cost evaluated in the BFS associated to B

vector of reduced costs

gives

$$x_B + B^{-1}F\,x_F = B^{-1}b$$

Test of optimality (tableau form). For a "max" [resp. "min"] problem, all elements in the row 0 and columns $x_{F,1}, \dots, x_{F,n-m}$ must be ≤ 0 [resp. ≥ 0]

Ex. Product mix

$$\max\ c^T x \qquad x^T = \begin{bmatrix} x_1 & x_2 & s_1 & s_2 & s_3 \end{bmatrix} \quad c^T = \begin{bmatrix} 30 & 20 & 0 & 0 & 0 \end{bmatrix}$$

$$Ax = b$$

$$x \geq 0$$

$$A = \begin{bmatrix} 8 & 4 & 1 & 0 & 0 \\ 4 & 6 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 640 \\ 540 \\ 100 \end{bmatrix}$$

Starting basis for phase 2: $B = \begin{bmatrix} A_1 & A_4 & A_5 \end{bmatrix} = \begin{bmatrix} 8 & 0 & 0 \\ 4 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

Initial simplex tableau

		x_1	x_2	s_1	s_2	s_3
	0	30	20	0	0	0
s_1	640	8	4	1	0	0
s_2	540	4	6	0	1	0
s_3	100	1	1	0	0	1

→ s_2 and S_3 are already dependent variables. To put the tableau in the canonical form w.r.t. B, x_1 must enter the basis and s_1 must leave the basis

Pivot operation

		x_1	x_2	s_1	s_2	s_3	
	0	30	20	0	0	0	$-[2400 \quad 30 \quad 15 \quad \frac{15}{4} \quad 0 \quad 0]$
s_1	640	8	4	1	0	0	
s_2	540	4	6	0	1	0	$-[320 \quad 4 \quad 2 \quad \frac{1}{2} \quad 0 \quad 0]$
s_3	100	1	1	0	0	1	$-[80 \quad 1 \quad \frac{1}{2} \quad \frac{1}{8} \quad 0 \quad 0]$

$$Aux = [\ 80 \quad 1 \quad \tfrac{1}{2} \quad \tfrac{1}{8} \quad 0 \quad 0\ ]$$

Tableau in the canonical form w.r.t. B

		x_1	x_2	s_1	s_2	s_3
	-2400	0	5	$-\frac{15}{4}$	0	0
x_1	80	1	$\frac{1}{2}$	$\frac{1}{8}$	0	0
s_2	220	0	4	$-\frac{1}{2}$	1	0
s_3	20	0	$\frac{1}{2}$	$-\frac{1}{8}$	0	1

Set $x_B^T = [x_1 \;\; s_2 \;\; s_3]$ and $x_F^T = [x_2 \;\; s_1]$. One reads

$$\bar{x}_B = B^{-1}b = \begin{bmatrix} 80 \\ 220 \\ 20 \end{bmatrix} \qquad B^{-1}F = \begin{bmatrix} \frac{1}{2} & \frac{1}{8} \\ 4 & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{8} \end{bmatrix}$$

Is B optimal? $r_F^T = [5 \;\; -\frac{15}{4}] \not\geq 0 \rightarrow$ the basis is not optimal. In particular, increasing x_2 will increase the cost

Computation of a new BFS that improves the cost and test of unboundedness

B: current feasible basis. We assume $r_F \not\leq 0$ ($r_F \not\geq 0$ for "min" problems)

Then, $\exists j : r_{F,j} > 0$ ($r_{F,j} < 0$ for "min" problems)

↳ j-th element of r_F

Rmk. If $x_{F,j}$ becomes > 0, the cost increases (decreases for "min" problems)

Bland's rule (part I). If there is more than one index j such that $r_{F,j} > 0$, pick the one associated to the variable x_i with minimal index i

Problem: how much $x_{F,j}$ can increase?

Assume $x_{F,j}$ is in the column q of the tableau. Up to a permutation of the columns one has

$$
\begin{array}{cc|ccc:c:c:c}
 & & x_{B,1} & \cdots & x_{B,m} & \cdots & x_{F,j} & \cdots \\
 & & 0 & \cdots & 0 & * & \bar{c}_q & * \\
\hline
x_{B,1} & \bar{b}_1 & 1 & & 0 & & \bar{a}_{1,q} & \\
\vdots & \vdots & & \ddots & & * & \vdots & * \\
x_{B,m} & \bar{b}_m & 0 & & 1 & & \bar{a}_{m,q} &
\end{array}
$$

$*$ = uninteresting blocks

Keeping $x_{F,i} = 0 \ \forall i \neq j$ one has ← ASSUMPTION

$$
\left.
\begin{aligned}
x_{B,1} &= -x_{F,j}\,\bar{a}_{1,q} + \bar{b}_1 \\
&\ \vdots \\
x_{B,m} &= -x_{F,j}\,\bar{a}_{m,q} + \bar{b}_m
\end{aligned}
\right\}
$$

Note that all uninteresting coeffs have been multiplied by zero

Feasibility of x implies $x_{B,i} \geq 0$, $i = 1, \dots, m$. Hence

$$\begin{aligned} -x_{F,j}\, \bar{a}_{1,q} + \bar{b}_1 &\geq 0 \\ &\vdots \\ -x_{F,j}\, \bar{a}_{m,q} + \bar{b}_m &\geq 0 \end{aligned}$$

Case I (unboundedness). All $\bar{a}_{1,q}, \dots, \bar{a}_{m,q}$ are ≤ 0. Since $x_{F,j}$ can grow arbitrarily, the LP problem is unbounded

Case II (find the most restrictive constraint). Each inequality with $\bar{a}_{i,q} > 0$ puts an upper bound to $x_{F,j}$

Let

$$\frac{\bar{b}_p}{\bar{a}_{p,q}} = \min_{i:\, \bar{a}_{i,q} > 0} \frac{\bar{b}_i}{\bar{a}_{i,q}}$$

Bland's rule (part II). If there are multiple indices i for which the minimum ratio is achieved, pick up the one corresponding to the basic variable x_i with minimal index i

The most restrictive inequality is

$$x_{B,p} = -x_{F,s}\,\bar{a}_{p,q} + \bar{b}_p \geq 0$$

Setting $x_{F,j} = \dfrac{\bar{b}_p}{\bar{a}_{p,q}}$, then

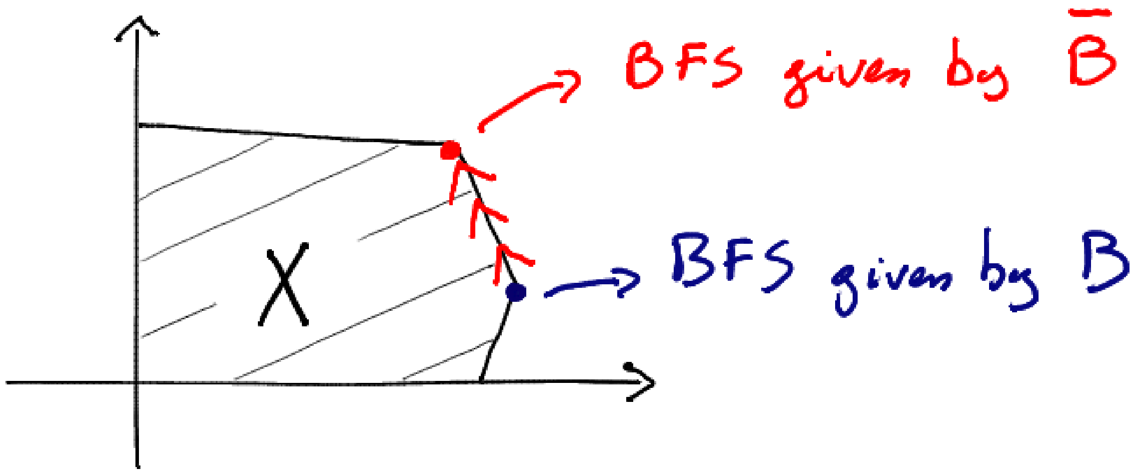
- the cost is increased by $\bar{c}_q \cdot \dfrac{\bar{b}_p}{\bar{a}_{p,q}}$ → increment in $x_{F,j}$ from zero
- $x_{B,p}$ becomes zero

Therefore, $x_{F,j}$ enters the basis and $x_{B,p}$ leaves the basis → a single pivoting on the element in the pivot row $x_{B,p}$ and pivot column $x_{F,j}$.

Rmk. The minimal positive (≥ 0) ratio defines the BV that leaves the basis.

The new basis $\bar{B}$ is obtained by replacing the column of B associated to $x_{B,p}$ with the column A_q. Problem: is $\bar{B}$ feasible?

Lemma. $\bar{B}$ is a feasible basis. If the vertices associated to B and $\bar{B}$ are different, they are adjacent



Ex. Product mix (ctd)

$$\max c^T x \qquad x^T = [x_1 \;\; x_2 \;\; s_1 \;\; s_2 \;\; s_3] \qquad c^T = [30 \;\; 20 \;\; 0 \;\; 0 \;\; 0]$$
$$Ax = b$$
$$x \geq 0$$

$$A = \begin{bmatrix} 8 & 4 & 1 & 0 & 0 \\ 4 & 6 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 640 \\ 540 \\ 100 \end{bmatrix}$$

Starting basis for phase 2: $B = [A_1 \;\; A_4 \;\; A_5]$

Tableau in the canonical form w.r.t. B $\longrightarrow$

		x_1	x_2	s_1	s_2	s_3
	-2400	0	5	$-\frac{15}{4}$	0	0
x_1	80	1	$\frac{1}{2}$	$\frac{1}{8}$	0	0
s_2	220	0	4	$-\frac{1}{2}$	1	0
s_3	20	0	$\frac{1}{2}$	$-\frac{1}{8}$	0	1

Phase 2 - Iteration 1

		x_1	x_2	s_1	s_2	s_3
	-2400	0	5	$-\frac{15}{4}$	0	0
x_1	80	1	$\frac{1}{2}$	$\frac{1}{8}$	0	0
s_2	220	0	4	$-\frac{1}{2}$	1	0
s_3	20	0	$\frac{1}{2}$	$-\frac{1}{8}$	0	1

Optimality test. $r_F^T = [5 \quad -\frac{15}{4}] \not\leq 0 \rightarrow$ The BFS is not optimal

Choice of the NBV that enters the basis

x_3 with associated reduced cost > 0 and minimum index J

↳ x_2 becomes basic $\longrightarrow$ pivot column = column "x_2"

		x_1	x_2	s_1	s_2	s_3
	-2400	0	5	$-\frac{15}{4}$	0	0
x_1	80	1	$\frac{1}{2}$	$\frac{1}{8}$	0	0
s_2	220	0	4	$-\frac{1}{2}$	1	0
s_3	20	0	$\frac{1}{2}$	$-\frac{1}{8}$	0	1

Choice of the BV that leaves the basis and test of unboundedness

Ratios:

$$\frac{80}{\frac{1}{2}} = 160$$

$$\frac{220}{4} = 55$$

$$\frac{20}{\frac{1}{2}} = 40 \rightarrow$$ minimal positive ratio. s_3 leaves the basis and defines the pivot row

Pivoting on the pivot element (at the intersection of the pivot row and column)

		x_1	x_2	s_1	s_2	s_3
	-2400	0	5	$-\frac{15}{4}$	0	0
x_1	80	1	$\frac{1}{2}$	$\frac{1}{8}$	0	0
s_2	220	0	4	$-\frac{1}{2}$	1	0
s_3	20	0	$\frac{1}{2}$	$-\frac{1}{8}$	0	1

AUX = $[40 \quad 0 \quad 1 \quad -\frac{1}{4} \quad 0 \quad 2]$

$-[200 \quad 0 \quad 5 \quad -\frac{5}{4} \quad 0 \quad 10]$

$-[20 \quad 0 \quad \frac{1}{2} \quad -\frac{1}{8} \quad 0 \quad 1]$

$-[160 \quad 0 \quad 4 \quad -1 \quad 0 \quad 8]$

		x_1	x_2	s_1	s_2	s_3
	-2600	0	0	$-\frac{5}{2}$	0	-10
x_1	60	1	0	$\frac{1}{4}$	0	-1
s_2	60	0	0	$\frac{1}{2}$	1	-8
x_2	40	0	1	$-\frac{1}{4}$	0	2

Phase 2 - Iteration 2

		x_1	x_2	s_1	s_2	s_3
	-2600	0	0	$-\frac{5}{2}$	0	-10
x_1	60	1	0	$\frac{1}{4}$	0	-1
s_2	60	0	0	$\frac{1}{2}$	1	-8
x_2	40	0	1	$-\frac{1}{4}$	0	2

Optimality test. $r_F^T = [-\frac{5}{2} \quad -10] \leq 0 \rightarrow$ The BFS is optimal and the algorithm stops.

Reading the results from the final tableau

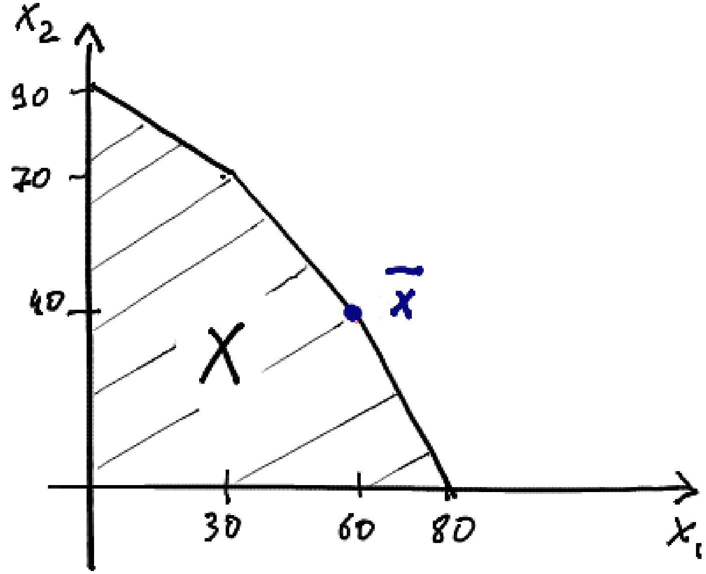
		x_1	x_2	s_1	s_2	s_3
	-2600	0	0	$-\frac{5}{2}$	0	-10
x_1	60	1	0	$\frac{1}{4}$	0	-1
s_2	60	0	0	$\frac{1}{2}$	1	-8
x_2	40	0	1	$-\frac{1}{4}$	0	2

Optimal solution $\bar{x}$ given by

$$\bar{x}_B^T = [x_1 \; s_2 \; x_2] = [60 \; 60 \; 40]$$

$$\bar{x}_F^T = [s_1 \; s_3] = [0 \; 0]$$

Optimal cost : 2,600



Ex. @ home

Consider the LP problem

$$\max \; x_1 + x_2$$
$$6x_1 + 4x_2 + x_3 = 24$$
$$3x_1 - 2x_2 + x_4 = 6$$
$$x_1, \ldots, x_4 \geq 0$$

Verify that the basis associated to the variables x_1 and x_2 is feasible and, starting from this basis, solve the problem running phase 2 of the simplex algorithm.